

QI 2009 Worksheets 1 and 2: some worked examples

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I. WORKSHEET 1 QUESTION 3

This is a personal view on how I would answer some of the first parts of question 3 (the numbering system was encrypted, so I can only refer to the parts in an approximate manner), plus a lot of background information. Before I start a stampede, this IN NO WAY reflects what I expected from you as an answer!! It is merely intended to fill in some of the knowledge gaps which I noticed in some of the submissions.

(Note that this is not very different from the tutorial session we had a few weeks ago).

A. Notation

For standard things like $\mathbf{U}(2)$, $\mathbf{O}(3)$ etc I will not give explicit definitions - the best quick reference these days is probably Wikipedia ...

Our starting point for the purposes of this discussion will always be a *normalized pure state*

$$|\psi\rangle = \cos\frac{\alpha}{2}|0\rangle + e^{i\phi}\sin\frac{\alpha}{2}|1\rangle \in \mathbb{C}^2,$$

for some $0 \leq \alpha < 2\pi$ and $0 \leq \phi < 2\pi$, which we will consider as corresponding to a density matrix $\rho = \rho_{|\psi\rangle}$ defined by:

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| \\ &= \begin{pmatrix} \cos^2\frac{\alpha}{2} & e^{-i\phi}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ e^{i\phi}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix} \quad (\text{by definition of outer product}) \\ &= \frac{1}{2} \begin{pmatrix} 1 + \cos\alpha & \cos\phi\sin\alpha - i\sin\phi\sin\alpha \\ \cos\phi\sin\alpha + i\sin\phi\sin\alpha & 1 - \cos\alpha \end{pmatrix} \quad (\text{using standard trigonometric identities}) \\ &= \frac{1}{2} (\mathbb{1} + (\cos\phi\sin\alpha)\sigma_X + (\sin\phi\sin\alpha)\sigma_Y + (\cos\alpha)\sigma_Z) \quad (\text{grouping together in the basis } \{\mathbb{1}, \text{Paulis}\}) \\ &= \frac{1}{2} (\mathbb{1} + \vec{\mathbf{r}}_\rho \cdot \vec{\sigma}), \end{aligned}$$

where $\mathbb{1}$ denotes the 2x2 identity matrix, $\vec{\sigma}$ is the 3-element 'vector' (of 2x2 matrices!) whose 'components' consist of the usual Pauli matrices σ_X, σ_Y and σ_Z , and $\vec{\mathbf{r}}_\rho \in \mathbb{R}^3$ is the (unit) Bloch vector

$$\vec{\mathbf{r}}_\rho = (\cos\phi\sin\alpha, \sin\phi\sin\alpha, \cos\alpha)$$

corresponding to ρ in the Bloch sphere.

Even if I forget to say it occasionally, $|\psi\rangle$ and $\rho = \rho_{|\psi\rangle}$ will always refer to the same state in this section.

B. Unitary evolution in \mathbb{C}^2 versus rotations in \mathbb{R}^3

So take an arbitrary pure state $|\psi\rangle$ as above. The gist of question 3 of the worksheet was the correspondence in the 1-qubit case between the following three situations:

1. The unitary evolution of the pure state $|\psi\rangle$, which we model in a complex Hilbert space \mathcal{H} of dimension 2 (ie \mathbb{C}^2 equipped with the usual inner product). The operators are elements U of the matrix group $U(2)$ which act by the usual linear action of 2x2 complex matrices on vectors in \mathcal{H} (ie left multiplication):

$$|\psi\rangle \mapsto U|\psi\rangle.$$

2. The unitary evolution of the same pure state $\rho = |\psi\rangle\langle\psi|$ considered this time as a *positive trace 1 density matrix* living in 2x2 complex matrix space $M_{2 \times 2}(\mathbb{C})$. Since ρ is pure it satisfies the stronger condition that $\text{Tr}\rho^2 = 1$. The unitary matrices U are as above: they cause evolution of $|\psi\rangle$ to $U|\psi\rangle$ and therefore of the adjoints $\langle\psi|$ to $\langle\psi|U^\dagger$. Hence the unitary group $U(2)$ acts via *conjugation* on the state ρ :

$$(\rho =) |\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger (= U\rho U^\dagger).$$

3. The rotation in \mathbb{R}^3 of the real Bloch vector \vec{r}_ρ corresponding to that same pure state ρ , under the action of operators which ‘look like’ the unitaries in $U(2)$. We need to find an orthogonal matrix O_U in the group $O(3)$ which acts on \vec{r}_ρ like U did on ρ . Here once again $O(3)$ acts via the usual linear action of 3x3 real operators on vectors in \mathbb{R}^3 (ie again just left multiplication):

$$\vec{r}_\rho \mapsto O_U \vec{r}_\rho.$$

Let $\mathbf{Pure}_2 \subset M_{2 \times 2}(\mathbb{C})$ denote the set of all equivalence classes of 2x2 density matrices corresponding to pure states in our 1-qubit Hilbert space \mathcal{H} (see [7]). Note that the elements of \mathbf{Pure}_2 may be taken to be represented by rank 1, unit trace, positive (therefore Hermitian) matrices. Let \mathbf{S}^2 denote as usual the unit sphere inside \mathbb{R}^3 , which in our situation will represent all of the pure states (again, up to equivalence) in \mathcal{H} . The construction of the Bloch sphere gives us essentially a 1-1 correspondence between these sets (see for example [3], [4] §1.2 and [8]):

$$\left\{ \begin{array}{l} \text{unit vectors } |\psi\rangle \in \mathcal{H} \\ \text{up to global phase} \end{array} \right\} \longleftrightarrow \mathbf{Pure}_2 \longleftrightarrow \mathbf{S}^2$$

given by

$$|\psi\rangle \longleftrightarrow \rho = \rho_\psi \longleftrightarrow \vec{r}_\rho$$

As an aside (thanks to Terry) I should point out that the geometric object constructed out of the ‘elements’ of the top left-hand side (ie equivalence classes of complex points in \mathbb{C}^2) is known mathematically as *1-dimensional complex projective space*, denoted $\mathbb{C}\mathbb{P}^1$, and that this beautiful geometric approach to the study of unitary evolution and state space in general has itself spawned a vast and rich literature. A wonderful introduction to the whole field (although the index is hopeless so you need to search a bit) is [1]. This is also by the way a good place to find out about convexity/simplices/etc which appeared near the end of question 3 on worksheet 1.

This establishes the framework for moving from any one of the sets of vectors in cases 1, 2 or 3 to the others. We now need to look at how unitary evolution is performed in each setup. Between case 1 and case 2 we made the correspondence explicit in the definition; so it remains only to see how the unitary evolution of density matrices in 2 is translated into that of real (Bloch) vectors in 3, and vice-versa. This requires that we define how to go from the unitary operator U to the orthogonal operator O_U , and further that we show that with such a definition we end up with the same physical picture whether we perform it in the context of the Hermitian matrices under U , or that of the Bloch sphere under O_U .

Just to summarize pictorially: in the language of algebra, the above correspondence between cases 2 and 3 is the same as saying that the following diagram *commutes*:

$$\begin{array}{ccc} \mathbf{Pure}_2 & \begin{array}{c} \rho \mapsto \vec{r}_\rho \\ \longrightarrow \end{array} & \mathbf{S}^2 \\ U \downarrow & & \downarrow O_U \\ \mathbf{Pure}_2 & \begin{array}{c} \longrightarrow \\ U\rho U^\dagger \mapsto \vec{r}_{U\rho U^\dagger} \end{array} & \mathbf{S}^2 \end{array}$$

So we start on the upper left-hand side with our pure state ρ , and move to the lower right-hand side. If we go around the diagram clockwise then we first find the Bloch vector $\vec{\mathbf{r}}_\rho \in \mathbb{R}^3$ of ρ , then map that (3-dimensional real) vector under the action of O_U to $O_U \vec{\mathbf{r}}_\rho$. If instead we go around the diagram anti-clockwise, then we first make the 2x2 unitary U act on ρ , taking it to $U\rho U^\dagger$, and then take the Bloch vector of this new pure state, namely $\vec{\mathbf{r}}_{U\rho U^\dagger}$. The statement that the diagram commutes is then equivalent to saying that we get the same answer no matter which way we choose to go around: that is to say,

$$\vec{\mathbf{r}}_{U\rho U^\dagger} = O_U \vec{\mathbf{r}}_\rho. \quad (1)$$

The mathematics of this correspondence is straightforward if you assume lots of things, and tricky if you start from scratch. I am going to take a bit of a middle road, assuming lots of things but giving plausible references from readily-available books. Then we will prove the particular central result from the worksheet about rotations around the y -axis using these preliminaries.

1. Unitary ‘rotations’

The strategy is to find a nice concise representation of the unitary matrices $U \in \mathbf{U}(2)$ in such a way that they are readily identifiable with their counterparts $O_U \in \mathbf{O}(3)$. The natural place to start is some sort of generalized ‘rotation’ matrices since rotations are easily visualized in the Bloch sphere.

It is not too much work to establish (see for example equation (4.12) in [4] and the ‘proofs’ (ie exercises ...) around it; or else §2.4 of [2] or any other text on 2x2 unitary matrices) that every 2x2 unitary matrix U may be represented (non-uniquely) by a matrix of the form:

$$U = \begin{pmatrix} \cos \alpha e^{i(\beta+\gamma+\delta)} & -\sin \alpha e^{i(\beta+\gamma-\delta)} \\ \sin \alpha e^{i(\beta-\gamma+\delta)} & \cos \alpha e^{i(\beta-\gamma-\delta)} \end{pmatrix}, \quad (2)$$

where $\alpha, \beta, \gamma, \delta$ are real angles which are determined up to an integral multiple of 2π . (To prove this formula for U , just begin by writing U as a general complex 2x2 matrix of the form: $U = \begin{pmatrix} r_a e^{i\theta_a} & r_b e^{i\theta_b} \\ r_c e^{i\theta_c} & r_d e^{i\theta_d} \end{pmatrix}$ where the $r_i \geq 0$ are real and the θ_i represent angles in $[0, 2\pi)$. Then use the fact that $UU^\dagger = \mathbb{1}$ etc.)

By some judicious variable-changing - literally just halving a few angles and playing with minus signs (again see §4.2 of [4]) - (2) may be manipulated into a form which reveals it to be expressible as (say)

$$U = e^{i\phi} \mathbf{R}_z(\epsilon) \mathbf{R}_y(\zeta) \mathbf{R}_z(\eta),$$

where ϕ is a global phase and ϵ, ζ and η are (real!) angles, and where the rotation matrices

$$\begin{aligned} \mathbf{R}_x(\theta) &= e^{-i\frac{\theta}{2}\sigma_x} = \begin{bmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}, \\ \mathbf{R}_y(\theta) &= e^{-i\frac{\theta}{2}\sigma_y} = \begin{bmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix}, \\ \mathbf{R}_z(\theta) &= e^{-i\frac{\theta}{2}\sigma_z} = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \end{aligned}$$

are the standard ones defined in the lectures/statement of the question. So we have arrived at a point where we know that *any* 2x2 unitary matrix $U \in \mathbf{U}(2)$ may be expressed (up to a global phase) as a product of these specific rotation matrices $\mathbf{R}_x, \mathbf{R}_y, \mathbf{R}_z$. Conversely, these matrices \mathbf{R}_x etc are easily shown to be unitary from their definitions. (Note that we have seen the decomposition into y, z -rotations but we could take any pair out of the triple x, y, z . Indeed any pair of linearly independent unit axes in \mathbb{R}^3 will serve to afford a decomposition of the form in (2) although one must be careful of the range of the angles η etc. involved (ie they cannot be restricted to $[0, 2\pi)$)).

If we now consider how these unitaries act on our given state $\rho = \rho_{|\psi\rangle}$, we are in situation (2) from the initial cases above. Namely, U acts by conjugation on ρ : recall a pure state $|\psi\rangle$ is mapped to $U|\psi\rangle$, hence ρ maps to

$$U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger. \quad (3)$$

So having broken up the unitary into the global phase factor - which under the adjoint outer product operation gives a factor of $e^{i\phi}e^{-i\phi} = 1$ and so disappears - times separate rotation matrices \mathbf{R}_z etc: in order to determine how unitaries in general act upon ρ , we need only study the evolution under each \mathbf{R} .

So that I don't spend my life writing this out, I am going to focus now simply on the case where θ is an arbitrary angle and the 'axis of rotation' (bearing in mind this is 2-dimensional complex Hilbert space and so the nomenclature is only that which is carried over from the analogous situation in the Bloch sphere in \mathbb{R}^3) is the y -axis (as I did in the tutorial).

So we begin as always with our chosen pure state $\rho = \rho_{|\psi\rangle}$. Our unitary evolution is described then by the matrix $\mathbf{R}_y(\theta)$, so according to (3) ρ evolves to:

$$\mathbf{R}_y(\theta)\rho\mathbf{R}_y(\theta)^\dagger. \quad (4)$$

Writing this out explicitly as matrices, and using the decomposition in (1) to express ρ via its Bloch vector

$$\vec{\mathbf{r}}_\rho = (r_x, r_y, r_z) = (\cos\phi \sin\alpha, \sin\phi \sin\alpha, \cos\alpha),$$

we obtain

$$\begin{aligned} \mathbf{R}_y(\theta)\rho\mathbf{R}_y(\theta)^\dagger &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \cdot \frac{1}{2} \cdot (\mathbb{1} + r_x\sigma_X + r_y\sigma_Y + r_z\sigma_Z) \cdot \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\ &= \frac{1}{2} \cdot \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \cdot \begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

which we may laboriously multiply out, or else notice that since $\mathbf{R}_y(\theta)$ is unitary:

$$\mathbf{R}_y(\theta)\mathbb{1}\mathbf{R}_y(\theta)^\dagger = \mathbf{R}_y(\theta)\mathbf{R}_y(\theta)^\dagger = \mathbb{1},$$

and further for each of the σ_i matrices (using the trig relations for $\theta/2 \leftrightarrow \theta$):

$$\mathbf{R}_y(\theta)\sigma_X\mathbf{R}_y(\theta)^\dagger = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix},$$

$$\mathbf{R}_y(\theta)\sigma_Y\mathbf{R}_y(\theta)^\dagger = Y$$

$$\mathbf{R}_y(\theta)\sigma_Z\mathbf{R}_y(\theta)^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

hence the above simplifies to

$$\begin{aligned} \mathbf{R}_y(\theta)\rho\mathbf{R}_y(\theta)^\dagger &= \frac{1}{2} \cdot \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \cdot (\mathbb{1} + r_x\sigma_X + r_y\sigma_Y + r_z\sigma_Z) \cdot \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\ &= \frac{1}{2} \cdot \left(\mathbb{1} + r_x \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} + r_y\sigma_Y + r_z \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \right) \\ &= \frac{1}{2} \cdot (\mathbb{1} + (r_x \cos\theta + r_z \sin\theta)\sigma_X + r_y\sigma_Y + (r_z \cos\theta - r_x \sin\theta)\sigma_Z). \end{aligned}$$

That is to say, the Bloch vector of the state to which ρ evolves under the conjugate action of $\mathbf{R}_y(\theta)$ is

$$\vec{\mathbf{r}}_{\mathbf{R}_y(\theta)\rho\mathbf{R}_y(\theta)^\dagger} = (r_x \cos\theta + r_z \sin\theta, r_y, r_z \cos\theta - r_x \sin\theta). \quad (5)$$

To refer back to the notation of (1), our U is $\mathbf{R}_y(\theta)$ and we must now find explicitly an O_U such that $\vec{\mathbf{r}}_{U\rho U^\dagger} = O_U\vec{\mathbf{r}}_\rho$.

2. Real rotations

Now all but one of the points on the surface of the Bloch sphere may be uniquely represented using (real) spherical coordinates $P = (1, \alpha, \phi)$ where 1 represents the fixed radius, α is the so-called *polar angle* (ie the positive angle with the z -axis, analogous to the latitude, so $0 \leq \alpha \leq \pi$) and ϕ is the so-called *azimuthal angle*, the positive angle ($0 \leq \phi < 2\pi$) with the x -axis and the analogue of the longitude. The 'South Pole' is not unique in that the azimuthal angle is indeterminate, but it is the only ill-defined point under this coordinate system. Of course, we may also use the standard Cartesian system $P = (x, y, z)$ to represent the same points and we fix axes in the usual way so that

there are inverse mappings from one system to the other via (loosely - to set it all up properly is very quirky and is not relevant here):

$$(r, \alpha, \phi) \longleftrightarrow (r \sin \alpha \cos \phi, r \sin \alpha \sin \phi, r \cos \alpha) \quad (6)$$

and

$$(x, y, z) \longleftrightarrow (\sqrt{x^2 + y^2 + z^2}, \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \arctan \frac{y}{x}). \quad (7)$$

Sticking for a second with the Cartesian coordinates, it is well-known (see for example [5] §6.2, or [6]) that given any ‘axis’ $\vec{\mathbf{a}} = (a_x, a_y, a_z)$, which we shall for simplicity take to be a unit vector, the rotation of \mathbb{R}^3 around $\vec{\mathbf{a}}$ through an arbitrary angle θ (all relative to the usual ‘right-hand rule’) is given by the matrix (in $\mathbf{SO}(3)$):

$$\Gamma_{\vec{\mathbf{a}}, \theta} = e^{-i\theta(\vec{\mathbf{a}} \cdot \vec{\mathbf{X}})}, \quad (8)$$

where analogously to the 2x2-Hermitian case with the Pauli matrices, we define a vector of matrices

$$\vec{\mathbf{X}} = (X, Y, Z)$$

of ‘infinitesimal generators’ of the rotation group where each matrix X, Y and Z is in turn defined as

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

Notice that the matrix $-i\theta(\vec{\mathbf{a}} \cdot \vec{\mathbf{X}})$ which is the eventual exponent above, analogously to the complex case, is *skew-symmetric* and ‘*traceless*’ (ie of trace zero). This has the consequence that the resulting rotation $\Gamma_{\vec{\mathbf{a}}, \theta}$ is respectively orthogonal and of determinant 1 (ie a so-called *proper rotation*).

The matrix for a given rotation $\Gamma_{\vec{\mathbf{a}}, \theta}$ is a little unwieldy at first sight ...

$$\Gamma_{\vec{\mathbf{a}}, \theta} = \begin{pmatrix} a_x^2 + (1 - a_x^2) \cos \theta & a_x a_y (1 - \cos \theta) - a_z \sin \theta & a_x a_z (1 - \cos \theta) + a_y \sin \theta \\ a_x a_y (1 - \cos \theta) + a_z \sin \theta & a_y^2 + (1 - a_y^2) \cos \theta & a_y a_z (1 - \cos \theta) - a_x \sin \theta \\ a_x a_z (1 - \cos \theta) - a_y \sin \theta & a_y a_z (1 - \cos \theta) + a_x \sin \theta & a_z^2 + (1 - a_z^2) \cos \theta \end{pmatrix} \quad (10)$$

So we have a form for a 3-dimensional real rotation around any given axis, of any given angle.

3. Equating the unitary and real rotations in the case of the y-axis

The claim now is that in the notation of (1) again, we may choose O_U to be the expression in (10) where $\vec{\mathbf{a}}$ is the y-axis, and the angle is θ . So it just remains to show that the Bloch vector which we obtained in (5) is the same as we would obtain were we to rotate the Bloch vector $\vec{\mathbf{r}}_\rho = (r_x, r_y, r_z)$ by an angle of θ radians around the y-axis (ie where our axis $\vec{\mathbf{a}} = (0, 1, 0)$). Substituting $a_x = 0, a_y = 1, a_z = 0$ into (10) we obtain the matrix:

$$O_U = O_{\mathbf{R}_y(\theta)} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \quad (11)$$

which, when applied to the Bloch vector (r_x, r_y, r_z) gives us precisely the expression in (5). \square

II. WORKSHEET 2 QUESTION 5(III)

This was the question about ‘unitary remixing’. Although it was kind-of optional whether you chose to do it or not, there was a part which asked you to prove that two sets of Kraus operators yield exactly the same evolution of a given state ρ provided that they are related by a unitary matrix in the given way.

That is to say, given that

$$\tilde{K}_i = \sum_j V_{ij} K_j$$

where V is a unitary matrix, and the K_j and \tilde{K}_i are two sets of Kraus operators, we need to show that

$$\sum_i \tilde{K}_i \rho \tilde{K}_i^\dagger = \sum_j K_j \rho K_j^\dagger.$$

Now this is not hard, but it is tricky, because He-who-set-the-question wrote V_{ij} and not V_{ji} : so most of you who did attempt it kind-of shut your eyes and pretended it was V_{ji} and hoped at my advanced age I wouldn't either have the attention span or the eyesight to read the little subscripts.

But unfortunately I did.

Here is the solution to the question as it was actually written, which no-one seemed to get ...

I denote by M^T the transpose of a matrix M , by a^* the complex conjugate of a scalar a , and by $M^\dagger = (M^T)^* = (M^*)^T$ the conjugate transpose of M . (Note that in this context, $(V_{il}M)^\dagger \neq V_{li}^* M^\dagger$ as several people wrote: V_{il} is a *scalar* here which happens also to be a coefficient of an n -by- n matrix, but as a scalar its 'conjugate transpose' is just its complex conjugate ... ie V_{il}^*).

$$\begin{aligned} \sum_i \tilde{K}_i \rho \tilde{K}_i^\dagger &= \sum_i \left(\sum_j V_{ij} K_j \right) \rho \left(\sum_l V_{il} K_l \right)^\dagger \\ &= \sum_{i,j,l} V_{ij} K_j \rho V_{il}^* K_l^\dagger \\ &= \sum_{i,j,l} V_{ij} V_{il}^* K_j \rho K_l^\dagger \\ &= \sum_{i,j,l} V_{ji}^T V_{il}^* K_j \rho K_l^\dagger \\ &= \sum_{i,j,l} V_{ji}^T (V^T)_{il}^\dagger K_j \rho K_l^\dagger \\ &= \sum_{j,l} \left(\sum_i V_{ji}^T (V^T)_{il}^\dagger \right) K_j \rho K_l^\dagger. \end{aligned}$$

This is almost what we want, because if U is a unitary matrix then $\sum_i U_{ji} U_{il}^\dagger = \delta_{jl}$, where δ_{ab} is the Kronecker delta-function. But we are given that V is unitary, hence $VV^\dagger = V^\dagger V = \mathbb{1}$, hence

$$V^T (V^T)^\dagger = (V^\dagger)^* V^* = (V^\dagger V)^* = \mathbb{1}^* = \mathbb{1}.$$

So V^T is also unitary. So continuing where we left off:

$$\begin{aligned} \sum_i \tilde{K}_i \rho \tilde{K}_i^\dagger &= \sum_{j,l} \left(\sum_i V_{ji}^T (V^T)_{il}^\dagger \right) K_j \rho K_l^\dagger \\ &= \sum_{j,l} \delta_{jl} K_j \rho K_l^\dagger \\ &= \sum_j K_j \rho K_j^\dagger \end{aligned}$$

as required.

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- [1] Ingemar Bengtsson and Karol Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* Cambridge University Press 2008.
[2] Stephen M. Barnett, *Quantum Information*. Oxford University Press 2009.
[3] http://en.wikipedia.org/wiki/Bloch_sphere
[4] M.A. Nielsen, I.L. Chuang, *Quantum computation and quantum information*. Cambridge University Press 2000.

- [5] H.F. Jones *Groups, Representations and Physics*. Taylor & Francis 1998.
- [6] <http://www.euclideanspace.com/maths/algebra/matrix/orthogonal/rotation/index.htm>
- [7] I am being a bit vague here. The equivalence relation I am referring to is of course that between *rank-one* unit trace positive matrices under the equivalence relations induced by normalization and the ‘cancelling’ of global phases; however one also needs to be careful about the ‘south pole’ of the Bloch sphere ... sort-of like the usual ‘hairy ball’ problem from topology ...
- [8] in the case of *mixed* states one extends the actions described here by linearity but the equivalence relation is more difficult. However we end up with a 1-1 correspondence between the interior of the Bloch ball, and the rank-*two* unit trace positive matrices. More later, maybe ...